INFINITE HORIZON OPTIMAL CONTROL ON MANIFOLDS

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ABSTRACT. This paper gives a brief outline of a program of formulating (infinite horizon) optimal control problems as restricted variational problems on manifolds. Based on a generalized Lagrange multiplier theorem we shall indicate how the formalism may be used to obtain results about optimal feedback control of infinite horizon optimal control problems. Moreover, connections with recent work about exterior differential systems and variational problems are given.

INTRODUCTION. This paper is concerned with a variational approach to optimal control. A *variational problem* can be given by a Lagrangian function $L(x,\dot{x},t)$, an end cost function $h(x)$, a time interval $I = [0,T]$ or $[0,\infty)$, an initial point and a target set S. Then one wants to minimize the action

$$J(x) = \int_I L(x(t),\dot{x}(t),t)dt + h(x(T))$$

over all possible curves $\{x(t), t \in I\}$, with $x(0) = x_0$, $x(T) \in S$. The problem is called *restricted* if the curves to be considered are restricted in some sense. We shall use the following abbreviations:

CEFHVP: *clamped end point, finite horizon variational problem*; here the target set consists of one point and we have no end cost function; moreover we have $I = [0,T]$;

FEFHVP: *free end point, finite horizon variational problem*; here the target set is some, at least one-dimensional, connected subset of the state space of $I = [0,T]$;

CEIHVP (FEIHVP): *clamped end point (free end point) infinite horizon variational problem*; $I = [0,\infty)$.

If certain properties are not specified we delete the indicating capitals. Optimal control problems can be considered to be variational problems in the space of states and inputs, with curves restricted to be state-input trajectories of a given system. We shall use the same abbreviations as for variational problems except that the V is deleted (IHP means infinite horizon optimal control problem).

In section 1 we shall give definitions and results for finite horizon variational problems. We restrict attention to first order conditions, although higher order conditions can be treated in a similar way. Curves satisfying the first order conditions are called stationary. The Lagrange multiplier rule given in this section relates stationary curves of a restricted problem, satisfying certain conditions, with stationary curves of an unrestricted problem on a higher dimensional manifold. The latter can be solved using a well-known Cartan characterization. In section 2 these results are extended to infinite horizon problems. In section 3 we show that an optimal control problem can be formulated as a variational problem. Application of the Lagrange multiplier theorem then yields an intrinsic coordinate-free characterization of stationary curves, which under certain regularity conditions comes down to the familiar Euler-Lagrange or Hamiltonian formalism if written in local coordinates. The general formalism can be used to analyze optimal feedback control of general infinite horizon optimal control problems.

In this paper we deleted details and proofs of theorems, these can be found in Bus [2] and [3]. For unexplained notations also see Bus [2,3] and Spivak [5].

1. THE FINITE HORIZON VARIATIONAL PROBLEM. Let M be a smooth (C^∞) manifold of dimension n. Denote $I=[0,T]$ $(T\in\mathbb{R}_+)$ to be the time interval. The *target set* S is assumed to be either one point $x_T\in M$ (CEFHVP), or a smooth connected submanifold of M of dimension $s\geq 1$ (FEFHVP). Instead of giving a Lagrangian $L(x,\dot{x},t)$ we give, more generally, an *action* 1-*form* α, which is a smooth differential 1-form on M. $h:S\to\mathbb{R}$ denotes the *end cost function* and $x_0\in M$ is the given *initial point*. The variational problem so defined is denoted by $VP(M,\alpha,h,S)$.

Given a smooth injective curve $\phi:I\to M$ with $\phi(0)=x_0$, $\phi(T)\in S$, we can give the following definitions.

DEFINITION 1.1. *An injective map* $\tilde{\phi}\in C^\infty((-\delta,\delta)\times I,M)$ $(\delta>0)$ *is called a* *__variation of__* ϕ *for the* FHVP *iff*

(i) $\tilde{\phi}(0,t)=\phi(t)$, $\forall t\in I$;

(ii) $\tilde{\phi}(\varepsilon,0)=\phi(0)$, $\tilde{\phi}(\varepsilon,T)\in S$, $\forall\varepsilon\in(-\delta,\delta)$.

(We denote: $\phi_\varepsilon(\cdot)=\tilde{\phi}(\varepsilon,\cdot)$.)

DEFINITION 1.2. *An injective curve* $\phi\in C^\infty(I,M)$ *is* *__stationary__* *for* $VP(M,\alpha,h,S)$ *if* $\phi(0)=x_0$, $\phi(T)\in S$ *and the first order condition holds:*

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\left[h(\phi_\varepsilon(T))+\int_I\phi_\varepsilon^*\alpha\right]=0 \quad , \tag{1.1}$$

for all variations ϕ_ε *of* ϕ.

It appears that we can restrict attention to variations which satisfy $\tilde{\phi}(\varepsilon,t) = \phi(t)$ on some neighborhood of $t=0$ (similar for CE conditions).

One might consider variations induced by vector fields along the curve ϕ and define stationarity in terms of Lie derivatives with respect to such vector fields. Although such a definition is extremely useful, particularly to ease many proofs, we shall avoid it here for the sake of brevity. For details we refer to [2] and [3].

For a given 2-form ω on M we define the *Cartan system* $C(\omega)$ to be the (Pfaffian) differential ideal generated by the set of 1-forms

$$\{X \lrcorner\, \omega \mid X \in X(M)\}$$

(cf. [4], where $X(M)$ denotes the set of smooth vector fields on M and $\lrcorner$ denotes contraction: $(X \lrcorner\, \omega)(Y) = \omega(X,Y), \ \forall Y \in X(M)$). Then we can give the following crucial characterization of stationary curves due to Cartan (see [2]).

PROPOSITION 1.3. *An injective curve* $\phi \in C^\infty(I,M)$ *is stationary for* $VP(M,\alpha,h,S)$ *if and only if*

(i) $\phi(I)$ *is an integral manifold of* $C(d\alpha)$;
(ii) $\phi(0) = x_0$, $\phi(T) \in S$;
(iii) $(dh+\alpha)|_S(\phi(T)) \equiv 0$ (*transversality*) .

We may introduce restrictions on curves in M via smooth (C^∞) codistributions of fixed dimension on M. It is shown in [2] that the condition that curves are trajectories of a given (nonlinear) system can easily be expressed in such a way. We say that an injective curve $\phi \in C^\infty(I,M)$ is *admissible* under restriction codistribution E if $\phi^*\beta = 0$ for all $\beta \in E$. Moreover, we call ϕ *stationary under restriction* E if it is stationary with respect to all admissible variations. However, the following more restrictive concept appears to be more useful.

DEFINITION 1.4. *An injective curve* $\phi \in C^\infty(I,M)$ *is* *<u>formally stationary</u>* *under restriction codistribution* E *(smooth and of fixed dimension) for the* FHVP *if* $\phi(0) = x_0$, $\phi(T) \in S$, ϕ *is admissible under* E *and*

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \phi_\varepsilon^*\beta = 0, \quad \forall\beta \in D \Rightarrow \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \left(h(\phi_\varepsilon(T)) + \int_I \phi_\varepsilon^*\alpha\right) = 0 \quad , \tag{1.2}$$

for all variations ϕ_ε *of* ϕ.

It is easily seen that formal stationarity implies stationarity. However, the converse is in general not true. We say that if stationarity implies formal stationarity for a certain variational problem, then this variational problem is *normal*.

Now let $\pi : T^*M \to M$ be natural projection. Then the *canonical* 1-*form* θ on T^*M is defined by

$$\theta(\zeta)(v) = \zeta(\pi_* v), \ \forall \zeta \in T^*M, \quad v \in T_\zeta T^*M \quad . \tag{1.3}$$

And the *Cartan form* on E associated with a 1-form α on M is defined by

$$\theta_\alpha = \pi_E^* \alpha + \theta_E \quad . \tag{1.4}$$

Note that E is a subbundle of T^*M, the subscript E denotes restriction to E. With these preliminaries we can state the Lagrange multiplier theorem. Here it is not just a rule whose satisfaction is a necessary condition for optimality, as it occurs usually in literature, but it gives necessary and sufficient conditions for formal stationarity of a curve (which in turn is a necessary condition for optimality).

THEOREM 1.5. (Lagrange multiplier theorem). *An injective curve* $\phi \in C^\infty(I,M)$ *is formally stationary for* FHVP(M,α,h,S) *under restriction codistribution* E, *if and only if there exists an injective* $\eta \in C^\infty(I,E)$ *with* $\phi = \pi_E \cdot \eta$, *which is stationary for the unrestricted* FHVP$(E,\theta_\alpha,h \cdot \pi_E,\chi(S))$, *where* χ *denotes some section* $\chi : M \to E$.

So a restricted VP can be "reduced" to an unrestricted VP on a higher dimensional manifold. For the latter we can use the characterization of Proposition 1.3. The fibre coordinates of the curve η in the vector subbundle $\pi_E : E \to M$ can be interpreted as the Lagrange multipliers in a given coordinate neighborhood.

2. THE INFINITE HORIZON PROBLEM. In this section we shall extend the results of section 1 to the IHVP. We have:

$$I = [0,\infty), \quad h \equiv 0.$$

With an IH-variation of $\phi : I \to M$ we mean a FE-variation of ϕ according to Definition 1.1 with $S = M$ (i.e., no condition on the end point). To assure finiteness of the cost integral we have to adapt the definition of stationarity in the following sense.

DEFINITION 2.1. *An injective curve* $\phi : I \to M$ *is*

(i) *stationary with respect to the* IHVP *with action form* α *iff*

$$\int_I |\phi^*\alpha| < \infty \tag{2.1}$$

(note that $\phi^*\alpha$ *is of the form* $\psi(t)dt$, *so that* $|\phi^*\alpha|$ *here means* $|\psi(t)|dt$) *and for all* IH-*variations* ϕ_ε *of* ϕ:

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \int_I \phi_\varepsilon^*\alpha = 0 \quad ; \tag{2.2}$$

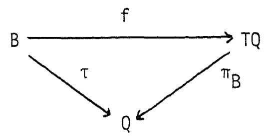
(ii) *formally stationary with respect to the restricted* IHVP *with action form* α *and restriction codistribution* E *iff* (2.1) *is satisfied and for all* IH-*variations* ϕ_ε *of* ϕ:

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \phi_\varepsilon^*\beta = 0, \quad \forall\beta \in E \Rightarrow \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \int_I \phi_\varepsilon^*\alpha = 0 \quad .$$

With these definitions the Lagrange multiplier theorem 1.5 is valid for infinite horizon problems without further restrictions, so that we do not reformulate it here.

3. THE OPTIMAL CONTROL PROBLEM. First we define the notion of a time-invariant nonlinear control system as introduced in Brockett [1] and Willems [6].

DEFINITION 3.1. *A nonlinear (time-invariant) control system* $\Sigma = \Sigma(Q,B,f)$ *is defined by smooth manifolds* Q *and* B, *a fibre bundle* $\tau : B \to Q$ *and a smooth map* $f : B \to TQ$ *such that the following diagram commutes*



We call $\Sigma(Q,B,f)$ *affine if* B *is a vector bundle and* f, *restricted to the fibres of* B, *is an affine map into the fibres of* TQ. Σ *is analytic if* B *and* Q *are analytic manifolds and* f *is an analytic map. We say that* $\psi : I \to Q$ *is a trajectory of* Σ *if* ψ *is absolutely continuous and*

$$\psi_*\left(\left.\frac{d}{dt}\right|_t\right) \in f(\tau^{-1}(\psi(t))) \quad \text{a.e. on } I \quad . \tag{3.1}$$

With a trajectory we associated a state-input trajectory $\zeta : I \to B$ *satisfying*

$$\tau(\zeta(t)) = \psi(t), \quad \psi_*\left(\left.\frac{d}{dt}\right|_t\right) = f(\zeta(t)), \quad \text{a.e. on } I \quad . \tag{3.2}$$

Note that the fibres of B represent the state dependent input spaces. If we choose local coordinates q for Q and u for the fibres $\tau^{-1}(q)$, then we obtain the familiar system equation $\dot{q} = f(q,u)$ (with abuse of notation for $f:(q,u) \to (q,f(q,u)))$. Then a trajectory $\psi: I \to Q$ is a solution of this differential equation for some given initial point and some associated input. The pair: trajectory with associated input, is the state-input trajectory and is in these coordinates often denoted by $\zeta(t) = (\psi(t)\nu(t))$. If Σ is affine then we can give a representation by

$$f(q,u) = f_0(q) + \sum_{i=1}^{m} u_i f_i(q) \quad , \tag{3.3}$$

with $u_i \in \mathbb{R}$, f_0, $f_i \in X(Q)$ $(i = 1,\dots,m)$. In the sequel we shall always assume that f is an injective immersion.

For brevity we shall only translate the infinite horizon stationary (optimal up to first order) control problem into a variational problem. The finite horizon problem is similar but requires some care with the target set. Moreover, the FHCEP does not always lead to a normal variational problem in the sense of section 1. So let be given: a nonlinear control system $\Sigma(Q,B,f)$, $I = [0,\infty)$, $h \equiv 0$, $S = Q$ and initial state q_0. Let $J^1(I,B)$ denote the 1-jet bundle of B, restricted to $I \subset \mathbb{R}$. Note that, given coordinates (q,u) on B, natural coordinates on $J^1(I,B)$ are $(q,u,\dot{q},\dot{u},t)$ and, moreover, $J^1(I,B) = TB \times I$. We assume that a cost function G is defined on $J^1(I,B)$. (At this moment there is no theoretical reason to restrict G to $B \times I$ as is usually done and we can imagine optimal control problems where the cost depends also on the derivatives.)

With every curve $\zeta \in C^\infty(I,B)$ we can naturally associate a curve $\zeta^p \in C^\infty(I,J^1(B,I))$ according to

$$t \to \left(\zeta_* \left(\frac{d}{dt}\Big|_t\right),t\right) \tag{3.4}$$

(in coordinates: $t \to (\zeta(t),\zeta(t),t)$). We consider the IHP of finding state-input trajectories $\zeta \in C^\infty(I,B)$ with $\tau \cdot \zeta(0) = q_0$ such that ζ is stationary for the cost

$$J(\zeta) = \int_I G(\zeta^p)dt \quad . \tag{3.5}$$

As admissible curves for the optimal control problem are state-input trajectories of the nonlinear system we may restrict the problem to the submanifold

$$M = \{(w,t) \in J^1(I,B) | f \cdot \pi(w,t) = \tau_*(w)\} \subset J^1(I,B) \quad . \tag{3.6}$$

Moreover the restriction that curves considered in M should be naturally associated to curves in B lead to a restriction codistribution E which is simply the canonical codistribution on $J^1(I,B)$ (see [4]) restricted to M. E is locally spanned by the 1-forms:

$$\begin{aligned} \beta_i &= dq_i - f_i(q,u)dt \ , \qquad i = 1,\dots,n \ , \\ \beta_j &= du_j - \dot{u}_j dt \ , \qquad j = 1,\dots,m \ , \end{aligned} \tag{3.7}$$

where coordinates (q,u) on B and the injective immersion f (locally given by components f_i, $i = 1,\dots,n$) induce natural coordinates $(q,u,\dot{u},t)$ on M. So the restricted variational problem associated with the given optimal control problem is: $VP(M,G_M dt,E)$, where $h \equiv 0$ and $S = \Omega$ are deleted and the subscript M denotes restriction to M. It can be proven that the variational problem thus associated with an infinite horizon, free end point optimal control problem is normal. This will in general not hold for clamped end point problems.

Application of the Lagrange multiplier rule yields the following result.

COROLLARY 3.2. *Assume that the variational problem associated with a given* IHP *is normal. Let* $\zeta \in C^\infty(I,B)$ *be injective and*

$$\int_T \left| G\left(\zeta_* \frac{d}{dt}\Big|_t\right), t\right) \Big| \, dt < \infty \quad .$$

Then ζ *is stationary for the* IHP *if and only if there exists an injective curve* $\eta \in C^\infty(I,E)$ *with* $i \cdot \pi_E \cdot \eta = \zeta^p$ *such that* η *is an integral curve of the Cartan system* $C(d\theta_G)$, *where* $\theta_G = \pi_E^\star(G_M dt) + \theta_E$ *is the Cartan form associated with* $G_M dt$, $i : M \to J^1(I,B)$ *the natural embedding,* $\pi : T^* M \to M$ *projection and the subscripts* M *and* E *denote restriction to* M *and* E.

So we see that stationary curves of the optimal control problem can be found as projections of integral curves of the exterior differential system $(C(d\theta_G),dt)$ on E. Note, moreover, that the distribution $(C(d\theta_G))^\perp$ which annihilates the Cartan system is typically not of constant dimension.

Griffiths [4, I.e. 1] gives a construction procedure to reduce $(C(d\theta_G),dt)$ on E to a system (I,dt) on $Y \subset E$, such that:

(i) the projection on Y of the set of integral elements of (I,dt) is surjective;

(ii) the integral manifolds of $(C(d\theta_G),dt)$ on E coincide with those of (I,dt) on Y.

In accordance with his terminology we call Y the *momentum space* and (I,dt) the *Euler-Lagrange differential system* associated with the optimal control problem. Then the following definition appears to be useful.

DEFINITION 3.3. *The optimal control problem is* *nonsingular* *if*

(i) $\dim Y = 2n+1$ $(\dim Q = n)$;

(ii) $(\theta_G|_Y) \wedge (d\theta_G|_Y)^n \neq 0$.

If an optimal control problem is nonsingular then we know by Darboux' theorem that we may choose local coordinates $Q_1,\dots,Q_n,P_1,\dots,P_n,T$ for Y such that (with summation over $i=1,\dots,n$)

$$\theta_G|_y = -H(Q,P,T)dT + P_i dQ_i \quad , \tag{3.8}$$

for some H, which is called the *Hamiltonian*. Exterior differentiation and contraction with $\partial/\partial Q_i$, $\partial/\partial P_i$ yields for (I,dt) on Y in this coordinates:

$$\begin{aligned} &dQ_i - H_{P_i} dT = 0 \quad , \\ &dP_i + H_{Q_i} dT = 0 \quad , \\ &dt \equiv F dT \quad \mathrm{mod}\{dQ_i, dP_i\} \ , \quad F \neq 0 \quad . \end{aligned} \tag{3.9}$$

Hence optimal curves should satisfy the Hamiltonian equations:

$$\frac{dQ_i}{dT} = \frac{\partial H}{\partial P_i} \quad , \quad \frac{dP_i}{dT} = -\frac{\partial H}{\partial Q_i} \quad . \tag{3.10}$$

We shall work out the procedure in natural coordinates $(q,u,\dot{u},t)$ for M to illustrate the ideas. Choose coordinates $(q,u,\dot{u},\lambda,\mu,t)$ for E ($e \in E$ can be written as $\lambda_i\beta_i(q,u,\dot{u},t) + \mu_j\beta_{n+j}(q,u,\dot{u},t)$ with β_k as in (3.7) and summation over i and j). Then we have

$$\theta_G = Hdt - \lambda_i dq_i - \mu_j du_j \quad ,$$

with

$$H(q,u,\dot{u},\lambda,\mu,t) = G(q,u,f(q,u),\dot{u},t) + \lambda^T f(q,u) + \mu^T u \quad .$$

Contracting $d\theta_G$ with $\partial/\partial q$, $\partial/\partial u$, $\partial/\partial\dot{u}$, $\partial/\partial\lambda$, $\partial/\partial\mu$ and equating to zero gives respectively:

$$
\begin{aligned}
&H_q dt + d\lambda = 0 \quad , \\
&H_\lambda dt - dq = 0 \quad , \\
&H_u dt + d\mu = 0 \quad , \\
&\dot{u}\ dt - du = 0 \quad , \\
&(G_u + \mu)dt = 0 \quad .
\end{aligned} \tag{3.11}
$$

If we assume that $G_{\dot{u}} = 0$ and $H_u = 0$ can be explicitly solved with respect to u (this holds e.g. if $G_{\dot{q}} = 0$, the system is affine and $\det G_{uu} \neq 0$) yielding $u = U(q,\lambda,t)$, then we can work out the reduction procedures yielding

$$
Y = \{e = (q,u,\dot{u},\lambda,\mu,t) \mid \mu = 0,\quad u = U(q,\lambda,t),\quad \dot{u} = \frac{d}{dt} U(q,\ ,t)\}
$$

$$
I : \begin{cases} H_q dt + d\lambda = 0 \quad , \\ \\ H_\lambda dt - dq = 0 \quad , \end{cases}
$$

with

$$
H = H(q,u,\lambda,t) = G(q,u,f(q,u),t) + \lambda^T f(q,u) \quad .
$$

Clearly the problem is nonsingular if $H \neq 0$, because $\dim Y = 2n+1$ and

$$
(\theta_G|_Y) \wedge (d\theta_G|_Y)^n = H dq \wedge d\lambda \wedge dt \neq 0 \quad .
$$

With the tools available now we might proceed in the following way to study existence and uniqueness of optimal feedback controls for IHP. First define

DEFINITION 3.4. *An* _*equilibrium curve*_ *of an* IHP *is an integral curve of the Euler-Lagrange system which has, in canonical coordinates, the form* $t \to (\overline{Q},\overline{R},T(t))$ *with* $\overline{Q}$ *and* $\overline{P}$ *constant.*

An equilibrium $y(t) = (\overline{Q},\overline{P},T(t))$ satisfies the equations:

$$
H_P(y(t)) = H_Q(y(t)) = 0 \quad .
$$

Then the *basin* $B \subset Y$ of an equilibrium $y(t)$ of the IHP is the set of all points in Y through which there goes an integral curve $t \to (Q(t),P(t),T(t))$ of the Euler-Lagrange system with $Q(t) \to Q$, $P(t) \to P$ for $t \to \infty$. Now let $\tilde{\pi} : Y \to Q$ be natural projection. Then existence of a stationary curve from $q_0 \in Q$ going to the equilibrium is assured if $q_0 \in \tilde{\pi}(B)$.

Finally, we shall illustrate the ideas by working out the linear-quadratic control problem.

$$
\begin{aligned}
&f(q,u) = Aq + Bu \quad , \\
&G(q,u) = q^T M q + u^T R u \quad ,
\end{aligned}
$$

where M and R are symmetric and R positive definite matrices. We as moreover that $(A,BR^{-1}B^T)$ is stabilizable. Then we obtain

$$Y = \{(q,u,\dot{u},\lambda,\mu,t)\,|\,\mu = 0,\quad u = -R^{-1}B^T\lambda,\quad \dot{u} = -R^{-1}B^T(Mq + A^T\lambda)\}$$

and the equilibrium is obtained as the solution of

$$H\begin{pmatrix} q \\ \lambda \end{pmatrix} = 0\ ,\quad \text{with}\quad H = \begin{pmatrix} A & -BR^{-1}B^T \\ -M & -A^T \end{pmatrix}$$

Hence $q = \lambda = 0$. The integral curves of the Euler-Lagrange system satisfy

$$\begin{pmatrix} \dot{q} \\ \dot{\lambda} \end{pmatrix} = H\begin{pmatrix} q \\ \lambda \end{pmatrix} \quad .$$

As we have stabilizability, there exists a stabilizing solution K^- of th algebraic Riccati equation

$$A^TK + KA - KBR^{-1}B^TK + M = 0 \quad .$$

The columns of $\begin{pmatrix} I \\ K^- \end{pmatrix}$ span an H-invariant subspace. Curves $(q(t),\lambda(t),t)$ satisfying

$$\dot{q} = (A - BR^{-1}B^TK^-)q \quad ,$$

$$\lambda = K^-q$$

are stable integral curves of the Eugler-Lagrange system on Y and the ba $\mathcal{B} \subset Y$ is given by

$$\mathcal{B} = \{(q,\lambda,t)\,|\,\lambda = K^-q\} \quad .$$

Hence $\tilde{\pi}(\mathcal{B}) = Q$ and the optimal feedback, given the initial point $q(0) =$ is obtained by computing the solution curve through $(q_0,K^-q_0,0)$ and usin the definition of Y to obtain u.

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